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HANANI TRIPLE SYSTEMS

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ABSTRACT

Hanani triple systems on $v \equiv 1 \pmod 6$ elements are Steiner triple systems having $(v-1)/2$ pairwise disjoint almost parallel classes (sets of pairwise disjoint triples that span $v-1$ elements), and the remaining triples form a partial parallel class. Hanani triple systems are one natural analogue of the Kirkman triple systems on $v \equiv 3 \pmod 6$ elements, which form the solution of the celebrated Kirkman schoolgirl problem. We prove that a Hanani triple system exists for all $v \equiv 1 \pmod 6$ except for $v \in \{7, 13\}$.

1. Background

In 1847, Kirkman [K] determined when, on a finite set V of $v = |V|$ *elements*, one can produce a collection $\mathcal{B}$ of 3-element subsets of V (called **triples** or **blocks**), with the property that every 2-element subset of V occurs as a subset of exactly one triple in $\mathcal{B}$. Kirkman proved that this is possible if and only if $v \equiv 1, 3 \pmod 6$. Such a configuration has come to be known as a **Steiner triple system of order** v, or STS(v).

Kirkman also studied the following more difficult problem:

> Is it possible to take fifteen schoolgirls for a walk on each of seven consecutive days, so that the girls walk three abreast, every girl walks every day, and no two girls walk in the same row twice?

In modern vernacular, this fifteen schoolgirl problem asks for an STS(15) with the additional property that the 35 triples can be partitioned into seven classes of five triples each, so that each class consists of (elementwise) disjoint triples. A set of pairwise disjoint triples that span all of the elements is a **parallel class** (PC) or **resolution class**, and a partition of the triples into parallel classes is a **resolution**.

Kirkman [K] generalized the fifteen schoolgirl problem by asking when a resolvable STS(v) exists, and provided some small examples of resolvable STS; these are now known as **Kirkman triple systems**. The existence of Kirkman triple systems, the well known **Kirkman schoolgirl problem**, was the subject of much study over the following one hundred and twenty years. In 1971, Ray-Chaudhuri and Wilson [RW] provided a complete solution, showing that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$. They also provide an interesting historical summary of the progress on the problem from Kirkman's time.

Kirkman triple systems have proved to be fundamental configurations in combinatorial design theory. Their uses in the construction of designs with block size four, and numerous classes of triple systems with additional properties, are far reaching. Hence, generalizations of KTS abound.

When $v \equiv 1 \pmod 6$, no STS(v) has a parallel class as $v \not\equiv 0 \pmod 3$. An **almost parallel class** (APC) for $v \equiv 1 \pmod 3$ is a set of $(v-1)/3$ pairwise disjoint triples, and an **almost resolution** is a partition of the triples into almost parallel classes. One might hope at first to find a partition of an STS(v) into almost parallel classes, but a disappointment is in store. An STS(v) has $v(v-1)/6$ triples, and each almost parallel class has $(v-1)/3$ triples. Thus we would require $v/2$ almost parallel classes, but v is odd. To circumvent this numerical problem and produce an analogue of Kirkman triple systems when $v \equiv 1 \pmod 6$, Kotzig and Rosa [KR] defined nearly Kirkman triple systems of order $6t$ in a manner equivalent to the following. For an STS(v), $v = 6t+1$, find $(v-1)/2$ almost parallel classes that are disjoint, and all miss a fixed element ∞. Deleting ∞ and the triples containing it, we have a resolution of a partial STS on $6t$ elements; this resolved partial STS is a **nearly Kirkman triple system** (NKTS). Kirkman triple systems have $(v-1)/2$ parallel classes, and the STS arising from nearly Kirkman triple systems have $(v-1)/2$ almost parallel classes all missing the same element.

In 1974, Hanani explored a different generalization. He examined collections of triples in which every 2-subset occurs exactly twice; these are **twofold triple systems**, and exist whenever $v \equiv 0, 1 \pmod 3$. Hanani [HA] established that there exists an almost resolvable twofold triple system if and only if $v \equiv 1 \pmod 3$, and a resolvable twofold triple system if and only if $v \equiv 0 \pmod 3$, $v \neq 6$.

These generalizations have found extensive applications in the construction of designs. Almost resolvable systems are most useful in the construction of designs with pairs occurring more than once. Nearly Kirkman triple systems are essentially resolutions of partial designs. If we extend such a system to the STS$(6t+1)$ defining it, and attempted at the same time to extend the resolution, each parallel class extends to an almost parallel class, but each of the $3t$ triples added in forming the STS shares an element. Let us define a **partial parallel class** (PPC) to be any set of pairwise disjoint triples. Then an NKTS$(6t)$ gives a partition of the STS$(6t+1)$ into $6t$ partial parallel classes, of which $3t$ are almost parallel classes and $3t$ contain just a single triple. In 1981, Phelps (private communication) asked for what orders it is possible to construct an STS(v), $v \equiv 1 \pmod 6$, having a partition into $(v+1)/2$ partial parallel classes, which is the minimum possible by numerical considerations. The minimum number of partial parallel classes required to partition the triples of any fixed system is the **chromatic index** χ' of the system. Hence Phelps's question asks for Steiner triple systems of orders congruent to $1 \pmod 6$ whose chromatic index is the minimum possible, $(v+1)/2$.

In addition to settling Phelps's question, our objective is to generalize the Kirkman schoolgirl problem in the strongest way. Hence we examine STS(v), $v \equiv 1 \pmod 6$, that have $(v-1)/2$ almost parallel classes, and the remaining $(v-1)/6$ triples form a partial parallel class. We call these STS **Hanani triple systems**, and denote them by HATS(v).

The relation of HATS to KTS and NKTS is clear, but they have in addition a close connection with Hanani's almost resolvable twofold triple systems. For suppose that $(V, \mathcal{B})$ is a HATS(v); each element is missed by exactly one of the partial parallel classes, so suppose that $\{1, \ldots, (v-1)/2\}$ are each missed by one of the almost parallel classes and that $\{(v+1)/2, \ldots, v\}$ are each missed by the remaining partial parallel class. Form a second HATS(v) $(V, \mathcal{B}')$ by renumbering the elements as $i \mapsto v + 1 - i$ for $1 \leq i \leq v$. Then $(V, \mathcal{B} \cup \mathcal{B}')$ is a twofold triple system; moreover, it is almost resolvable — form an almost resolution by taking all almost parallel classes of $\mathcal{B}$ and of $\mathcal{B}'$, and forming the final almost parallel class as the union of the partial parallel classes of $\mathcal{B}$ and $\mathcal{B}'$. Thus the existence of a Hanani triple system of order v provides not only a solution to Phelps's problem, but also a particularly elegant construction of an almost resolvable twofold triple system of order v, the designs studied by Hanani [HA].

In this paper, we prove:

MAIN THEOREM: *A Hanani triple system of order v exists if and only if $v \equiv 1 \pmod 6$, $v \notin \{7, 13\}$.*

Necessity is obvious from the definition, and the observation that the unique STS(7) has chromatic index 7, and both of the nonisomorphic STS(13) have chromatic index 8 [C].

2. Some direct constructions

Our first direct construction is a straightforward application of Bose's method of differences; for details of the method, see, e.g., Hall [H], and also [RW], Section 3.

Let $t \geq 1$ be an integer. An APC P on the set $V = \{\infty\} \cup Z_{6t+3} \times \{0, 1\}$ is **smooth** if the pure difference $\pm(2t+1)_0$ does not occur in P, and all other differences (pure or mixed) occur at most once in P.

It follows that in a smooth APC every difference other than the pure difference $\pm(2t+1)_0$ occurs exactly once, and the element not covered by the APC is x_0 for some $x \in Z_{6t+3}$. Without loss of generality, we may assume that any smooth APC contains the triple $\{\infty, 0_0, 0_1\}$.

Example 1: A smooth APC on 19 elements:
$\infty\ 0_0\ 0_1, 1_0\ 3_0\ 5_1, 2_0\ 7_0\ 8_1, 4_0\ 5_0\ 3_1, 8_0\ 2_1\ 4_1, 1_1\ 6_1\ 7_1$.

Example 2: A smooth APC on 31 elements:
$\infty\ 0_0\ 0_1, 1_0\ 2_0\ 4_0, 3_0\ 12_0\ 9_1, 6_0\ 10_0\ 11_1, 7_0\ 14_0\ 6_1, 8_0\ 1_1\ 4_1,$
$9_0\ 7_1\ 3_1, 11_0\ 13_1\ 14_1, 5_1\ 10_1\ 12_1, 13_0\ 2_1\ 8_1$.

LEMMA 2.1: *If there exists a smooth* APC *on V with $|V| = 12t + 7$ elements, then there exists a* HATS$(12t + 7)$.

Proof: Let $C = \{\{i_0, (i+2t+1)_0, (i+4t+2)_0\}: i = 0, 1, ..., 2t\}$. If P is a smooth APC, let, for $i \in Z_{6t+3}$,

$$P_i = \{\{i + a, i + b, i + c\}: \{a, b, c\} \in P\}.$$

Then $(V, C \cup \bigcup_{i \in Z_{6t+3}} P_i)$ is clearly an STS$(12t+7)$. Taking P_i, $i = 0, 1, ..., 6t+2$, and C as colour classes shows that this STS is a HATS. ∎

COROLLARY 2.2: *There exists a* HATS(v) *for* $v = 19, 31$.

Proof: See Examples 1 and 2. ∎

We can obtain a smooth APC (and thus a HATS) by a method similar to that which produces a Kirkman triple system from a Room square (cf. [MV]). Suppose we have a starter-adder constructed Room square R (cf. [DS]), with the underlying set $V = \{\infty\} \cup Z_{6t+3} \times \{1\}$, and assume w.l.o.g. that the upper lefthand corner cell of R contains the pair $\infty\ 0_1$. Label the $6t + 3$ columns of R with elements of $Z_{6t+3} \times \{0\}$ so that the ith column of R is labelled with i_0. Let $X_0 \subset Z_{6t+3} \times \{0\}$ be the set of labels of columns for which the first row cells in R are empty; clearly $|X_0| = 3t + 1$. If for some $y_0 \in X_0$ there exists a partition of $X_0 \backslash \{y_0\}$ into t triples $\{u_q, v_q, w_q\}, q = 1, ..., t$, such that $\{(u_q)_0, (v_q)_0, (w_q)_0\}, q = 1, ..., t$, together with $\{0_0, (2t+1)_0, (4t+2)_0\}$ is the set of base blocks of a cyclic STS$(6t+3)$ on $Z_{6t+3} \times \{0\}$ then

$$P = \{\{(u_q)_0, (v_q)_0, (w_q)_0\}: q = 1, ..., t\} \cup \{\{r_0, (x_r)_1, (y_r)_1\}: (x_r)_1, (y_r)_1 \text{ occurs in the first row of } R \text{ in the column labelled } r_1\}$$

is a smooth APC on $\{\infty\} \cup Z_{6t+3} \times \{0, 1\}$; the element not covered by P is y_0.

While the construction of [MV] produces a KTS$(12t+3)$ with maximum sub-STS$(6t+1)$, the situation is, in a certain sense, reversed here: we construct a HATS$(12t+7)$ with a maximum sub-STS$(6t+3)$. The smooth APC*s* in the next examples are obtained by the construction just described.

Example 3: A smooth APC on 43 elements:
$\infty\ 0_0\ 0_1, 6_0\ 7_1\ 14_1, 10_0\ 5_1\ 16_1, 12_0\ 2_1\ 19_1, 14_0\ 8_1\ 13_1, 15_0\ 3_1\ 18_1,$
$16_0\ 9_1\ 12_1, 17_0\ 6_1\ 15_1, 18_0\ 1_1\ 20_1, 19_0\ 10_1\ 11_1, 20_0\ 4_1\ 17_1, 1_0\ 2_0\ 11_0,$
$3_0\ 7_0\ 9_0, 5_0\ 8_0\ 13_0.$

Example 4: A smooth APC on 55 elements:
$\infty\ 0_0\ 0_1, 4_0\ 10_1\ 17_1, 5_0\ 4_1\ 23_1, 10_0\ 7_1\ 20_1, 11_0\ 5_1\ 22_1, 13_0\ 11_1\ 16_1,$
$16_0\ 3_1\ 24_1, 17_0\ 9_1\ 18_1, 20_0\ 2_1\ 25_1, 22_0\ 12_1\ 15_1, 23_0\ 8_1\ 19_1, 24_0\ 1_1\ 26_1,$
$25_0\ 13_1\ 14_1, 26_0\ 6_1\ 21_1, 1_0\ 15_0\ 21_0, 2_0\ 6_0\ 14_0, 7_0\ 9_0\ 12_0, 8_0\ 18_0\ 19_0.$

Example 5: A smooth APC on 67 elements:
$\infty\ 0_0\ 0_1, 7_0\ 10_1\ 23_1, 11_0\ 7_1\ 26_1, 12_0\ 9_1\ 24_1, 13_0\ 12_1\ 21_1, 14_0\ 6_1\ 27_1,$
$15_0\ 8_1\ 25_1, 17_0\ 5_1\ 28_1, 22_0\ 13_1\ 20_1, 23_0\ 4_1\ 29_1, 25_0\ 14_1\ 19_1, 26_0\ 2_1\ 31_1,$
$27_0\ 11_1\ 22_1, 28_0\ 15_1\ 18_1, 29_0\ 3_1\ 30_1, 30_0\ 1_1\ 32_1, 31_0\ 16_1\ 17_1, 1_0\ 5_0\ 10_0,$
$2_0\ 4_0\ 19_0, 6_0\ 9_0\ 16_0, 8_0\ 20_0\ 21_0, 18_0\ 24_0\ 32_0.$

Example 6: A smooth APC on 79 elements:
$\infty\ 0_0\ 0_1, 8_0\ 14_1\ 25_1, 9_0\ 12_1\ 27_1, 12_0\ 8_1\ 31_1, 13_0\ 6_1\ 33_1, 15_0\ 9_1\ 30_1,$
$16_0\ 13_1\ 26_1, 18_0\ 10_1\ 29_1, 20_0\ 5_1\ 34_1, 26_0\ 15_1\ 24_1, 27_0\ 4_1\ 35_1, 28_0\ 16_1\ 23_1,$
$31_0\ 17_1\ 22_1, 32_0\ 2_1\ 37_1, 33_0\ 7_1\ 32_1, 34_0\ 18_1\ 21_1, 35_0\ 3_1\ 36_1, 36_0\ 1_1\ 38_1,$
$37_0\ 19_1\ 20_1, 38_0\ 11_1\ 28_1, 1_0\ 11_0\ 22_0, 2_0\ 4_0\ 19_0, 3_0\ 6_0\ 10_0, 7_0\ 21_0\ 30_0,$
$5_0\ 17_0\ 25_0, 23_0\ 24_0\ 29_0.$

Example 7: A smooth APC on 91 elements:
$\infty\ 0_0\ 0_1, 5_0\ 19_1\ 41_1, 8_0\ 17_1\ 43_1, 12_0\ 22_1\ 38_1, 14_0\ 16_1\ 44_1, 17_0\ 21_1\ 39_1,$
$20_0\ 27_1\ 33_1, 21_0\ 18_1\ 42_1, 24_0\ 7_1\ 8_1, 25_0\ 24_1\ 36_1, 27_0\ 6_1\ 9_1, 28_0\ 20_1\ 40_1,$
$30_0\ 5_1\ 10_1, 32_0\ 28_1\ 32_1, 33_0\ 4_1\ 11_1, 34_0\ 23_1\ 37_1, 36_0\ 29_1\ 31_1, 37_0\ 25_1\ 35_1,$
$39_0\ 2_1\ 13_1, 40_0\ 26_1\ 34_1, 41_0\ 1_1\ 14_1, 42_0\ 3_1\ 12_1, 43_0\ 15_1\ 30_1, 2_0\ 19_0\ 23_0,$
$6_0\ 7_0\ 16_0, 9_0\ 11_0\ 29_0, 1_0\ 4_0\ 35_0, 13_0\ 18_0\ 26_0, 15_0\ 38_0\ 44_0, 3_0\ 10_0\ 22_0.$

Example 8: A smooth APC on 103 elements:
$\infty\ 0_0\ 0_1, 49_0\ 1_1\ 50_1, 47_0\ 2_1\ 49_1, 50_0\ 3_1\ 48_1, 48_0\ 4_1\ 47_1, 41_0\ 5_1\ 46_1,$
$37_0\ 6_1\ 45_1, 35_0\ 7_1\ 44_1, 46_0\ 8_1\ 43_1, 32_0\ 9_1\ 42_1, 29_0\ 10_1\ 41_1, 18_0\ 11_1\ 40_1,$
$21_0\ 12_1\ 39_1, 17_0\ 13_1\ 38_1, 20_0\ 14_1\ 37_1, 25_0\ 15_1\ 36_1, 2_0\ 16_1\ 35_1, 39_0\ 17_1\ 34_1,$
$45_0\ 18_1\ 33_1, 40_0\ 19_1\ 32_1, 44_0\ 20_1\ 31_1, 5_0\ 21_1\ 30_1, 3_0\ 22_1\ 29_1, 43_0\ 23_1\ 28_1,$
$38_0\ 24_1\ 27_1, 42_0\ 25_1\ 26_1, 11_0\ 12_0\ 36_0, 7_0\ 28_0\ 30_0, 19_0\ 22_0\ 33_0, 9_0\ 13_0\ 31_0,$
$1_0\ 6_0\ 16_0, 4_0\ 10_0\ 23_0, 8_0\ 15_0\ 24_0, 14_0\ 26_0\ 34_0.$

LEMMA 2.3: *There exists a* HATS(v) *for* $v \in \{43, 55, 67, 79, 91, 103\}$.

Proof: Take Examples 3–8 and Lemma 2.1. ∎

We have not produced yet any HATS of orders $\equiv 1 \pmod{12}$; all our examples constructed so far were of orders $v \equiv 7 \pmod{12}$. The next direct construction devised for orders $v \equiv 1 \pmod{12}$ is somewhat more involved.

Let $t \geq 1$ be an integer. An APC **P** on the set $V = \{\infty\} \cup Z_{6t} \times \{0,1\}$ is **greasy** if it has the following properties:

(i) $\mathbf{P} = T \cup H$ where
$H = \{\{\infty, 0_0, (3t)_0\}, \{0_1, a_i, b_j\}, \{(3t)_1, (a+3t)_i, (b+3t)_j\}\}$, $i, j \in \{0,1\}$,
and T is a set of $4t-3$ triples,

(ii) for $U = T \cup \{\{0_0, a_i, (b+3t)_j\}, \ \{0_1, a_i, b_j\}\}$, the pure differences $\pm(2t)_0$, $\pm(3t)_0$, $\pm(3t)_1$ do not occur in U but all other differences (pure or mixed) occur exactly once in U.

Lemma 2.4: *If there exists a greasy* APC *on V with $|V| = 12t+1$ elements, there exists a* HATS$(12t+1)$.

Proof: Let **P** be a greasy APC, with T, H, and U as defined above. Let $C = \{\{i_0, (i+2t)_0, (i+4t)_0\}: i = 0, 1, ..., 2t-1\}$, and

$$D_j = \{\infty, i_j, (i+3t)_j\}: i = 0, 1, ..., 3t-1, j = 0, 1$$

For $k \in Z_{6t}$, let $U_k = \{\{a+k, b+k, c+k\}: \{a,b,c\} \in U\}$, and let

$$\mathbf{B} = C \cup D_0 \cup D_1 \cup \bigcup_{k \in Z_{6t}} U_k.$$

Then clearly $(V, \mathbf{B})$ is an STS$(12t+1)$: the sets C, D_0 and D_1 use up precisely the differences not occurring in U. We have to show that $(V, \mathbf{B})$ is a HATS.

For $k = 0, 1, ..., 3t-1$, let $P_k = \{\{a+k, b+k, c+k\}: \{a,b,c\} \in \mathbf{P}\}$. Let $K = \{\{\infty, 0_1, (3t)_1\}, \ \{0_0, a_i, (b+3t)_j\}, \ \{(3t)_0, (a+3t)_i, b_j\}\}$, and let $\mathbf{Q} = T \cup K$; since **P** is an APC, so is **Q**. For $k = 3t, 3t+1, ..., 6t-1$, let $Q_k = \{\{a+k, b+k, c+k\}: \{a,b,c\} \in \mathbf{Q}\}$. Taking now P_k, $k = 0, 1, ..., 3t-1$, Q_k, $k = 3t, 3t+1, ..., 6t-1$, and C as the $6t+1$ colour classes shows that $(V, \mathbf{B})$ is a HATS. ∎

Corollary 2.5: *There exists* a HATS(25).

Proof: Take

$$H = \{\{\infty, 0_0, 6_0\}, \{0_1, 1_1, 3_1\}, \{6_1, 7_1, 9_1\}\},$$

$$T = \{\{5_1, 10_1, 10_0\}, \{5_0, 8_0, 4_1\}, \{2_0, 3_0, 8_1\}, \{7_0, 9_0, 11_1\}, \{11_0, 4_0, 2_1\}\};$$

then $K = \{\{\infty, 0_1, 6_1\}, \{0_0, 1_1, 9_1\}, \{6_0, 7_1, 3_1\}\}$. It is straightforward to verify that $\mathbf{P} = T \cup H$ is a greasy APC, and so Lemma 2.4 applies. ∎

COROLLARY 2.6: *There exists a* HATS(49).

Proof: Take $H = \{\{\infty, 0_0, 12_0\}, \{0_1, 1_1, 3_1\}, \{12_1, 13_1, 15_1\}\}$, and
$T = \{\{2_1, 6_1, 11_1\}, \{4_1, 10_1, 17_1\}, \{8_1, 16_1, 2_0\}, \{3_0, 4_0, 7_1\}, \{6_0, 8_0, 19_1\},$
$\{19_0, 22_0, 18_1\}, \{17_0, 21_0, 14_1\}, \{10_0, 15_0, 20_1\}, \{5_0, 11_0, 5_1\}, \{13_0, 20_0, 22_1\},$
$\{7_0, 16_0, 23_1\}, \{23_0, 9_0, 21_1\}, \{14_0, 1_0, 9_1\}\};$
then $K = \{\{\infty, 0_1, 12_1\}, \{0_0, 1_1, 15_1\}, \{12_0, 13_1, 3_1\}\}$. One verifies again directly that $\mathbf{P} = T \cup H$ is a greasy APC, and the corollary follows by Lemma 2.4. ∎

3. A recursive construction

Our basic induction argument is based on the following theorem.

THEOREM 3.1: *Let* $(V, \mathbf{G}, \mathbf{B})$ *be a group divisible design* (GDD) *with* $|V| = v$, $|G| \equiv 0 \pmod 3$ *for all groups* $G \in \mathbf{G}(|G| \geq 9)$, *and* $|B| \equiv 1 \pmod 3$ *for all blocks* $B \in \mathbf{B}$. *If for each* $G \in \mathbf{G}$ *there exists a* HATS$(2|G|+1)$ *then there exists a* HATS$(2v+1)$.

Proof: Put $V^* = V \times \{1,2\} \cup \{\infty\}$. For each $B \in \mathbf{B}$ let $V_B = B \times \{1,2\} \cup \{\infty\}$. Let (V_B, C_B, R_B) be a Kirkman triple system KTS$(2|B|+1)$ such that $B_a = \{\infty, a_1, a_2\} \in C_B$ for each $a \in B$. Each B_a is in a unique parallel class of R_B; let R_a^B be this parallel class. Further, for each group $G \in \mathbf{G}$ let (V_G, C_G), where $V_G = G \times \{1,2\} \cup \{\infty\}$, be a HATS$(2|G|+1)$ such that ∞ is not contained in a triple of the "short" colour class, say G^*, and $\{\infty, a_1, a_2\} \in C_G$ for each $a \in G$. Again, $\{\infty, a_1, a_2\}$ defines a unique almost parallel class R_a^G in C_G.

Define now $\mathbf{B}^* = \bigcup_{a \in V, B \in \mathbf{B}} R_a^B \cup \{G^* \colon G \in \mathbf{G}\}$, and

$$R_a^* = \bigcup_{B \in \mathbf{B}} R_a^B \cup \{R_a^G\} \text{ for } a \in V,\ R_G^* = \bigcup_{G \in \mathbf{G}} G^*.$$

It can be checked that $(V^*, \mathbf{B}^*)$ is a HATS$(2v+1)$ with colour classes R_a^*, $a \in V$, and the "short" colour class R_G^*. ∎

LEMMA 3.2: *If t and m are nonnegative integers such that $0 \leq t \leq m$ and there exists a set of three mutually orthogonal latin squares* (MOLS) *of order m then there exists a* GDD *with $12m + 3t$ elements, with blocks of size* 4 *and groups of size $3m$ and $3t$.*

Proof: If there exists a set of three MOLS(m) then there exists a GDD with blocks of size 5 and 5 groups of size m (cf. [BJL]; the GDD is actually a transversal design). Deleting $m - t$ elements from one of the groups yields a GDD $\mathbf{G}^*$ with blocks of size 4 and 5, and with 4 groups of size m and one group of size t. There exists a GDD G_1 with 4 groups of size 3 and blocks of size 4, and a GDD G_2 with 5 groups of size 3 and blocks of size 4; G_1 and G_2 are obtained by removing an element of $PG(2,3)$, and of $AG(2,4)$, respectively. Apply now Wilson's Fundamental Construction (cf. [BJL]) to $\mathbf{G}^*$ by giving each element a weight of 3 and using G_1, G_2 to obtain a GDD as in the statement of the lemma. ∎

If $t \geq 3$, all group sizes are at least 9 (and, of course, multiples of 3).

4. A few more constructions

Denote $M^* = \{v\colon \text{there exists a HATS}(v)\}$, and let $R = \{r\colon 2r + 1 \in M^*\}$. In this notation, we have already shown $\{9, 12, 15, 21, 24, 27, 33, 39, 45, 51\} \subset R$. The next few lemmas establish several further values as members of R.

LEMMA 4.1: $18 \in R$.

Proof: Put $V = \{\infty\} \cup Z_9 \times \{1,2,3,4\}$. Let $\mathbf{P}$ be the following APC (we omit brackets and commas):

$$\mathbf{P} = \{\infty\ 0_1\ 2_3, 1_1\ 1_2\ 4_3, 5_1\ 7_2\ 3_3, 4_2\ 5_3\ 4_4, 2_2\ 6_3\ 3_4, 3_1\ 7_3\ 0_4,$$
$$2_1\ 8_2\ 7_4, 7_1\ 8_1\ 6_2, 4_1\ 0_3\ 1_3, 6_1\ 1_4\ 5_4, 8_3\ 8_4\ 6_4, 3_2\ 5_2\ 0_2\},$$

and let $\mathbf{P}'$ be the APC obtained from $\mathbf{P}$ by adding 1 modulo 4 to the subscripts. Developing $\mathbf{P}$ and $\mathbf{P}'$, respectively, modulo 9 yields a total of 18 APC*s*; let $\mathbf{B}$ be the union of all triples of these 18 APC*s*. If $C = \{0_1\ 3_1\ 6_1, 0_4\ 3_4\ 6_4 \pmod 9\}$ is the "short" PPC then $(V, \mathbf{B} \cup C)$ is a HATS(37). ∎

LEMMA 4.2: $30 \in R$.

Proof: This is a similar construction to the one in the preceding lemma, but here $V = \{\infty\} \cup Z_{15} \times \{1,2,3,4\}$, $C = \{0_1\ 5_1\ 10_1, 0_4\ 5_4\ 10_4 \pmod{15}\}$,
$\mathbf{P} = \{\infty\ 0_1\ 10_3, 14_1\ 2_2\ 13_3, 12_1\ 4_2\ 14_3, 1_1\ 9_2\ 7_3, 11_1\ 13_2\ 1_4, 4_1\ 10_2\ 7_4,$
$13_1\ 12_2\ 8_4, 3_2\ 3_3\ 11_4, 5_2\ 6_3\ 5_4, 7_2\ 12_3\ 14_4, 14_2\ 11_3\ 3_4, 8_1\ 9_3\ 12_4, 3_1\ 8_3\ 2_4,$

$10_1\ 4_3\ 10_4, 2_1\ 0_3\ 4_4, 7_1\ 11_2\ 1_2, 5_1\ 6_1\ 9_1, 6_2\ 8_2\ 0_2, 1_3\ 2_3\ 5_3, 13_4\ 0_4\ 6_4\}$, and $\mathbf{P}$ and $\mathbf{P}'$ are developed modulo 15. The result is a HATS(61). ∎

LEMMA 4.3: $42 \in R$.

Proof: The construction that we use in this case is somewhat similar to that of Lemma 2.4 (using greasy APC*s*). Let $V = Z_{30} \times \{1,2\} \cup \{\infty, a_1, a_2, \ldots, a_{24}\}$. The HATS(85) $(V, \mathbf{B})$ to be constructed contains a sub-HATS(25) on $\{\infty, a_1, a_2, \ldots, a_{24}\}$. Let $H = \{\infty\ 0_1\ 15_1, 5_1\ 0_2\ 5_2, 20_1\ 15_2\ 20_2\}$, $K = \{\infty\ 0_2\ 15_2, 0_1\ 5_1\ 20_2, 15_1\ 20_1\ 5_2\}$, $T = \{1_1\ 11_1\ 3_2, a_1\ 28_1\ 29_2, a_2\ 21_1\ 27_2, a_3\ 14_1\ 24_2, a_4\ 2_1\ 16_2, a_5\ 25_1\ 14_2, a_6\ 13_1\ 9_2, a_7\ 29_1\ 2_2, a_8\ 19_1\ 26_2, a_9\ 12_1\ 23_2, a_{10}\ 6_1\ 22_2, a_{11}\ 22_1\ 13_2, a_{12}\ 10_1\ 7_2, a_{13}\ 27_1\ 1_2, a_{14}\ 26_1\ 4_2, a_{15}\ 7_1\ 19_2, a_{16}\ 24_1\ 11_2, a_{17}\ 17_1\ 10_2, a_{18}\ 8_1\ 6_2, a_{19}\ 23_1\ 28_2, a_{20}\ 16_1\ 25_2, a_{21}\ 4_1\ 17_2, a_{22}\ 3_1\ 21_2, a_{23}\ 18_1\ 12_2, a_{24}\ 9_1\ 8_2\}$. Let $C = \{0_2\ 10_2\ 20_2 \pmod{30}\}$, and let further $\mathbf{P} = H \cup T$, $\mathbf{P}' = K \cup T$; note that both $\mathbf{P}$ and $\mathbf{P}'$ are APC*s* while C is clearly a PPC. If $\mathbf{P}_i = \{\{x+i, y+i, z+i\} \colon \{x,y,z\} \in \mathbf{P}\}$, $\mathbf{P}'_i = \{\{x+i, y+i, z+i\} \colon \{x,y,z\} \in \mathbf{P}'\}$ where $a_j + i = a_j$ then $\mathbf{P}_i$, $i = 0, 1, \ldots, 14$, and $\mathbf{P}'_i$, $i = 15, 16, \ldots, 29$ are altogether 30 (disjoint) APC*s*. The elements missed by these 30 APC*s* are those of $Z_{30} \times \{2\}$. The pairs not contained in the union of our 30 APC*s* are those of a disjoint union of a K_{25} on $\{\infty, a_1, a_2, \ldots, a_{24}\}$, a $K_{6,6,6,6,6}$ on $Z_{30} \times \{1\}$, and a $K_{6,6,6,6,6}$ on $Z_{30} \times \{2\}$, together with those of C. To get further twelve APC*s*, proceed now as follows: there exists a resolvable GDD with 5 groups of size 6 having 12 *PCs* of triples [RS], and by Corollary 2.5, there exists a HATS(25). Construct now 12 APC*s* on $Z_{30} \times \{1,2\} \cup \{\infty, a_1, a_2, \ldots a_{24}\}$ by combining in the obvious way the 12 *PCs* of the GDD with the 12 APC*s* of the HATS(25). If C' is the "short" colour class of the HATS(25) then the triples of $C \cup C'$ together with the triples of the 42 APC*s* constructed above form a set of triples $\mathbf{B}$. It is easy to verify that $(V, \mathbf{B})$ is an STS(85); in fact, taking the 42 APC*s* as colour classes together with $C \cup C'$ as the 43rd colour class shows that it is a HATS(85). ∎

LEMMA 4.4: $54 \in R$.

Proof: By Theorem 4.5 of [S], there exists a Kirkman frame of type 18^6, i.e. a GDD with 6 groups of size 18 and blocks of size 3 with the property that the blocks can be partitioned into PPC*s* called **holey parallel classes** each of which misses the elements of precisely one group (for a definition of a Kirkman frame, see [S]). For each group $G_i, i = 1, 2, \dots, 6$, of this GDD, there are exactly 9 holey parallel classes missing G_i. Put now $V = \{\infty\} \cup \bigcup_{i=1} G_i$. By Lemma 2.1, there exists a HATS(19); 9 of its colour classes are APC*s*. Let, for $i = 1, 2, \dots, 6$, $(V_i, \mathbf{B}_i)$ be a HATS(19) where $V_i = \{\infty\} \cup G_i$, C_i is the 10th (short) colour class, and C_i does not cover ∞. For each $i = 1, 2, \dots, 6$, combine the 9 holey parallel classes missing the group G_i in the obvious way with the 9 colour classes of $(V_i, \mathbf{B}_i)$ which are APC*s* to obtain 9 APC*s* on V. Since we have 6 groups, this gives 54 APC*s*. Finally, take $\bigcup C_i$ as the short colour class. ∎

Observe that the construction of Theorem 3.1 is also, in effect, a Kirkman frame construction.

LEMMA 4.5: $\{66, 78\} \subset R$.

Proof: The proof is similar to that of Lemma 4.4. There exists a Kirkman frame of type $24^4 36^1$ [RS], and by Corollary 2.5 and Lemma 4.1, there exists a HATS(25) and a HATS(37). Adjoining a new element ∞ and "filling in" $G \cup \{\infty\}$ with a HATS(25) if G is a group of size 24, and with a HATS(37) if G is the group of size 36 yields a HATS(133). To show $78 \in R$, proceed in exactly the same way but start with a Kirkman frame of type $24^5 36^1$ which exists by [RS]. ∎

LEMMA 4.6: $87 \in R$.

Proof: There exists a GDD with 29 elements of type $4^6 5^1$ with blocks of size 4 and 5: Start with an $S(2, 4, 28)$ (see [BJL]) having two parallel classes C and C' such that $|C \cap C'| = 1$. Take the blocks of C as the groups and extend each of the blocks of C' with a new element ∞; $(C \cap C') \cup \{\infty\}$ will become the group of size 5. Now use Wilson's Fundamental Construction with GDD of type 3^4 and 3^5 to obtain a GDD on 87 points of type $12^6 15^1$ and block size four. Then apply Theorem 3.1. ∎

LEMMA 4.7: $138 \in R$.

Proof: There exists a GDD of type $4^{10} 6^1$ and with blocks of size 4 and 5 (start with a resolvable $S(2, 4, 40)$ (see [BJL]), and extend 6 of its parallel classes with 6 new infinite elements). Now proceed as in the proof of Lemma 4.6 to complete the construction. ∎

5. The main result

LEMMA 5.1: $R' = \{3s: 3 \leq s \leq 30\} \subset R$.

Proof: The direct constructions of Section 2 together with Lemmas 4.1–4.6 imply that we need to deal only with the values of $s = 12, 16, 19, 20, 21, 23, 24, 25, 27, 28, 30$. Since there exist $4-$GDDs of types 9^4, 12^4 and 9^9, applying Theorem 3.1 gives $36, 48, 81 \in R$. Similarly, the existence of a transversal design (for a definition see, e.g. [BJL]) $TD(7,9)$ and a $TD(10,9)$, respectively, together with an application of Theorem 3.1 yields $63, 90 \in R$. Taking $m = 4$, $t = 3$ in Lemma 3.2 gives $57 \in R$. Taking $m = 5$, and $t = 0, 3, 4, 5$, respectively, in Lemma 3.2 gives $60, 69, 72, 75 \in R$. Finally, taking $m = 7$ and $t = 0$ in Lemma 3.2 yields $84 \in R$. ∎

THEOREM 5.2: $R = \{3s: s \geq 3\}$. *(In other words, there exists a* HATS(v) *if and only if* $v \equiv 1 \pmod 6$, $v \geq 19$.*)*

Proof: In view of Lemma 5.1, we may assume $s \geq 31$. For every $s \geq 31$ there exist m and t such that $3s = 12m + 3t$ where $m \geq 7$, $t = 3, 4, 5$ or 6, and $t \leq m$. If there exists a set of 3 mutually orthogonal latin squares of order m then Theorem 3.1 applies, and $3s \in R$. A set of 3 MOLS(m) is known to exist for all values of m of interest to us (i.e. $m \geq 7$) except for $m = 10$ (cf. [BJL]). Thus we must still consider the values of $3s = 129, 132, 135$, and 138. But $138 \in R$ by Lemma 4.7, and taking in Lemma 3.2 $m = 9$ and $t = 7, 8$ and 9, respectively, gives $129, 132, 135 \in R$. This completes the proof. ∎

6. Conclusion

Even though the question posed by Phelps a decade ago is answered at last by our main theorem, there are several related problems that one might consider and that are largely open. One question related to the famous Erdös–Faber–Lovász conjecture is that about the *maximum* chromatic index of an STS(v). Colbourn and Colbourn [CC1] have shown that the chromatic index of a cyclic STS(v) cannot exceed v. More recently, Pippenger and Spencer [PS] used probabilistic methods to show that for sufficiently large v, the chromatic index of an STS(v) is asymptotic to $v/2$. A more ambitious question would be to ask about the spectrum (i.e. the range of possible values) for chromatic indices of STSs. Colbourn

([C]; see also [CC2]) has established that of the 80 STS(15)*s*, 4 have chromatic index 7, 13 have chromatic index 8, and the remaining 63 have chromatic index 9. Not much else seems to be known in this direction. In this connection, one may also inquire about the computational complexity of computing the chromatic index of STS*s* exactly (cf. [RC]).

A different avenue of generalization is to consider the minimum chromatic index of packings and coverings. For $v \equiv 0 \pmod 6$, the answer for packings is provided by NKTS; for coverings, see [AMS].

As another possible spinoff problem, one might ask about the minimum chromatic index of triple systems $TS(v, \lambda)$ with higher index . Hanani's results on the existence of resolvable $TS(v, 2)$ and almost resolvable $TS(v, 2)$ [HA] provide us with a complete answer for $\lambda = 2$: There exists a $TS(v, 2)$ with minimum chromatic index $v - 1$ if and only if $v \equiv 0$ or $1 \pmod 3, v \neq 6$ (the unique $TS(6, 2)$ has chromatic index 10).

However, to best of our knowledge, no such answer is available for $\lambda = 3$ and for $\lambda = 6$, the remaining "minimal" values of λ. Therefore let us conclude with the following two open problems.

Problem 1: Does there exist, for every $v \equiv 5 \pmod 6$, a $TS(v, 3)$ with minimum chromatic index $(3v + 5)/2$?

Problem 2: Does there exist, for every $v \equiv 2 \pmod 6$, a $TS(v, 6)$ with minimum chromatic index $3v + 4$?

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